**Sets and Logic - Solutions**

In Questions 1-4, S = {1, 2, 3, 4}; T = {2, 3, 4, 5}; U = {4, 6, 8}; V = {6, 7}.

**1.** S ∪ T = {1,2,3,4,5} --- the elements that are in S or T or both; note that elements only appear once in a set

**2.** S ∩ T = {2,3,4} --- the elements that are in both S and in T

**3.** S – T = {1} --- the elements that are in S, but are not in T

**4.** ((S - T) ∪ U) – V = ({1} ∪ U) – V = {1,4,6,8} – V = {1,4,8}

**5.** Which of the following sets is a *fuzzy* set, rather than a crisp set?

R is the set of real numbers

a. { x | x ∈ R; x > 0}

b. { x | x ∈ R; x2 < 0}

c. { x | x ∈ R; x is very large}

d. { x | x ∈ R; x is infinitely large}

e. None of the above

Clearly the answer is c, since the phrase “very large” is fuzzy. Note that in b and d the sets are simply empty … and therefore crisp.

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Consider the formula θ: (p ∧ q) → ¬q. We saw in the lectures that this formula is not a tautology – this can be demonstrated by finding an assignment of truth values (TRUE or FALSE) to the individual propositions (here, p and q) which makes θ false. In this case there was just one such assignment: p=TRUE and q=TRUE.

Recall also that a formula of the form ϕ → μ is false if and only if (iff) ϕ is true and μ is false.

**6.** Demonstrate that the formula θ below is not a tautology by finding an assignment of truth values (TRUE or FALSE) to its individual propositions which makes θ false.

θ: (p → q) → (q → p)

For θ to be false, the LHS must be true, and the RHS must be false. For the RHS to be false, q=TRUE and p=FALSE; fortunately this truth assignment makes the LHS true.

**7.** Demonstrate that the formula θ below is not a tautology by finding an assignment of truth values (TRUE or FALSE) to its individual propositions which makes θ false.

θ: ((¬p V q) ∧ (p ∨ ¬q)) → (p ∧ ¬q)

As above we need the LHS to be true, and the RHS to be false. For the RHS to be false, we need either p to be FALSE or q to be TRUE.

Let’s try p=FALSE, and see if we can find a truth assignment for q that makes the LHS true. For the LHS to be true, we need both ¬p V q and p ∨ ¬q to be true. If p=FALSE, then ¬p V q is true … good. If p=FALSE, then to make p ∨ ¬q true, we need that q=FALSE … and there is an answer: p=FALSE and q=FALSE

Suppose instead that we had tried q=TRUE. For the LHS to be true, we need both ¬p V q and p ∨ ¬q to be true. If q=TRUE, then ¬p V q is true … good. If q=TRUE, then to make p ∨ ¬q true, we need that p=TRUE … and there is another answer: p=TRUE and q=TRUE

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In the lectures we saw that in order to demonstrate that a formula θ is a tautology, it is necessary to demonstrate that there is no assignment of truth values (TRUE or

FALSE) to the individual propositions in θ which makes θ false.

In the case of a formula of the form ϕ → μ, we can demonstrate that the formula is a tautology by showing that there is no assignment of truth values that makes both ϕ true and μ false.

**8.** Demonstrate that the following formula is a tautology.

(p V q) ∧ (¬p V r) → (q ∨ r)

Suppose that the LHS is true, and the RHS is false, then q=FALSE and r=FALSE. If the LHS is true, then both p V q and ¬p V r must be true. For p V q to be true, we must have that p=TRUE (since q=FALSE), and for ¬p V r to be true, we must have the p=FALSE (since r=FALSE).

And this completes the proof: we simply can’t have the LHS being true, whilst the RHS is false, and the formula is therefore a tautology.

Alternatively you can always use a truth table: for a formula (as above) with 3 propositions, the full table would have 8 rows. In this case you could reduce it to just 2 rows since you are looking for a row (i.e., an assignment of truth values) in which the RHS is false (thus the two rows are (i) p=TRUE and q=FALSE and r=FALSE, and (ii) p=FALSE and q=FALSE and r=FALSE). As above you’d find that in both of these rows, the LHS is false (and hence the whole formula is true).